LESSON 3.2

The Merits of Polynomial Models

Polynomial models are quite versatile; so much so, in fact, that one must be careful about using them too often. This lesson demonstrates the versatility of polynomial models.

Activity 3.2

Table 3.4 gives data on a hypothetical object that has re-entered Earth's atmosphere. From these early observations, scientists must predict when the object will hit Earth.

Time (seconds)	Altitude (feet)
0	400,000
10	399,000
20	393,000
30	384,000

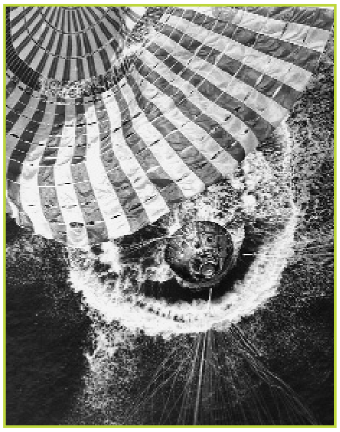
TABLE 3.4.
Falling body data.

Your objective in this activity is to model the data with several polynomial functions, and to evaluate their usefulness in predicting the time at which the object will reach Earth's surface. Here are a few points to remember as you work:

- Using a data-driven model to predict beyond the range of the data is risky. However, as this case demonstrates, sometimes there is no choice. Such predicaments emphasize the importance of explaining data-driven models.

- Polynomial functions are sensitive to slight changes in the values of the polynomial's coefficients. Maintain coefficients to as many digits of accuracy as possible (but keep the precision of the original data in mind when you use the model to make predictions).

- Models should be evaluated before they are used to make predictions. Check residuals for both size and pattern. When comparing two data-driven models, residuals alone do not determine which model is better, particularly if one model can be explained but not the other.



NASA

1. Find a quadratic regression model for the data in Table 3.4. Use it to predict when the object will reach the ground. Are you comfortable with the prediction?

2. Find a cubic (3rd-degree polynomial) regression model. Compare it to the quadratic model you found in Item 1. Which do you think is the better model? Explain.

3. Add a fifth pair (40, 375000) to Table 3.4. Compare quadratic, cubic, and quartic (4th-degree polynomial) regression models.

4. Summarize what you have learned in this activity about polynomial functions as models.

FITTING A POLYNOMIAL FUNCTION TO DATA

Polynomial regression is a technique that produces a polynomial function of given degree to fit a given set of data. For example, cubic regression produces a third-degree polynomial function for which the sum of the squares of the residuals is smaller than the corresponding sum for any other cubic. Many calculators do linear, quadratic, cubic, and quartic regression.

Fitting a Polynomial Exactly

Polynomial functions have a modeling capability possessed by no other simple function: They are capable of passing through every point in a set of data exactly. (Of course, a polynomial function, like any other function, cannot pass through two distinct points with the same x-coordinate.) If lack of error were the only criterion for modeling data, modelers would need no other functions.

In general, a set of n data pairs can be modeled with no error by a polynomial function of degree no more than $n - 1$. Exact fit polynomials can be found with the aid of a calculator program that implements the following mathematical procedure.

A function $f(x)$ captures data perfectly if every data point (a, b) satisfies the function; that is, if $f(a) = b$. (Such a function is called a **spline** for the data.) By substituting each data pair into a general polynomial function of degree one smaller than the number of data points, a system of equations is obtained. Since the number of equations matches the number of variables in the system, the system can be solved. The solution of the system gives the coefficients of the polynomial.

Example 2

Find the equation of a polynomial passing through each of the points (1, 7), (3, 14), (6, 2), and (8, 11). (Note that since this example uses four pairs, a calculator with a cubic regression feature can give the same result automatically. For simplicity, the number of pairs in this example has been kept small.)

Solution:

Since a third-degree polynomial has four coefficients (a, b, c, and d in $f(x) = ax^3 + bx^2 + cx + d$), substituting each of the four pairs into the general third-degree polynomial creates the system (1) of four equations with four unknowns:

$$(1)\quad \begin{aligned} 7 &= a(1)^3 + b(1)^2 + c(1) + d \\ 14 &= a(3)^3 + b(3)^2 + c(3) + d \\ 2 &= a(6)^3 + b(6)^2 + c(6) + d \\ 11 &= a(8)^3 + b(8)^2 + c(8) + d. \end{aligned}$$

Thus, the specific cubic can be found by solving the system:

$$(2)\quad \begin{aligned} 7 &= a + b + c + d \\ 14 &= 27a + 9b + 3c + d \\ 2 &= 216a + 36b + 6c + d \\ 11 &= 512a + 64b + 8c + d. \end{aligned}$$

There are a variety of methods for solving systems of equations. The symbolic method reduces the system to one with one less variable and one less equation, then repeats the process until a simple linear equation in one variable is obtained. If the first equation in (2) is subtracted from each of the others, the system becomes:

$$(3)\quad \begin{aligned} 7 &= 26a + 8b + 2c \\ -5 &= 215a + 35b + 5c \\ 4 &= 511a + 63b + 7c. \end{aligned}$$

System (3), in turn, can be reduced to a system with two equations and two variables by selecting one of the equations and adding a multiple of it to the second, then adding another multiple of the same equation to the third equation. If the proper multiples are chosen, the result is a system of two equations and two variables.

Multiplying the first equation by $\frac{-5}{2}$ and adding the result to the second eliminates the variable c in the second equation. Similarly, multiplying the first equation by $\frac{-7}{2}$ and adding the result to the third eliminates the variable c in the third equation. The system that results is:

$$(4)\quad \begin{aligned} -22.5 &= 150a + 15b \\ -20.5 &= 420a + 35b. \end{aligned}$$

Multiplying the first equation of system (4) by $-\frac{35}{15} = -\frac{7}{3}$ and adding it to the second equation eliminates b and reduces the system to a single linear equation in one variable, which can be solved:

$$32 = 70a$$

$$a = \frac{32}{70}.$$

Substituting this value into one of the two equations in the two-variable system (4) gives the value of b. The values of a and b can then be substituted into one of the three equations in system (3) to determine c, and a final substitution of the values of a, b, and c into one of the original four equations of system (2) determines d.

Needless to say, for a system of any size, the symbolic procedure is difficult to implement since the many multiplications and additions and the frequent occurrence of non-integer values make calculation errors likely.

Most calculator programs that solve systems do so by applying one of two matrix techniques that are discussed in detail in Chapter 7 of this book. For the purposes of this chapter, systems can be solved by using a calculator program.

Two TI-83 programs are provided for this lesson on the CD-ROM that accompanies this book. The program Polyfit finds a polynomial of degree $n - 1$ that captures n data points. The program Syssolv solves a system of n equations in n variables.

Application of the Polyfit program to the example above gives $a \approx 0.4571$, $b \approx -6.0714$, $c \approx 21.8429$, $d \approx -9.2286$. (If rounded at all, these coefficients should be maintained to several decimal places beyond the precision of the original data to ensure that predictions are accurate.) Thus the polynomial function is $y = 0.4571x^3 - 6.0714x^2 + 21.8429x - 9.2286$.

Checking the function against the original data is advisable. For example, evaluating for $x = 3$ gives 13.9992, or approximately 14.

Polynomial Precautions

Since polynomials can provide error-less models, it is tempting to apply them without thought. However, unless you can explain why a particular polynomial model is appropriate, be cautious: Although a polynomial function may capture the data perfectly, it may fail to capture the trend the data represent. **Figure 3.1** shows two ways in which a polynomial function of higher-degree may fail to capture a trend that a linear or quadratic function could capture well.

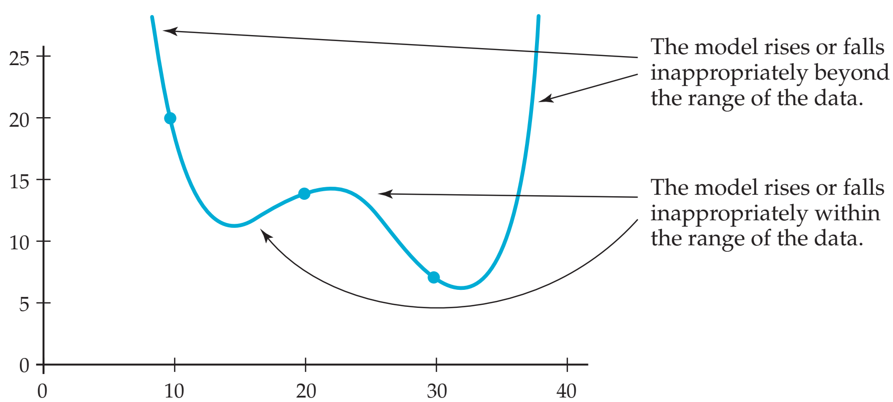


FIGURE 3.1. Failings of a polynomial model.

In situations in which a polynomial function is an appropriate model, care must be taken to avoid round-off error since predictions obtained from a polynomial model are quite sensitive to changes in the polynomial's coefficients. Therefore, it is advisable to maintain as many digits as possible in coefficients. Of course, predictions made with such a model should be rounded to the precision of the data used to generate the model.

Exercises 3.2

1. During the 1991 war in the Persian Gulf, American forces successfully used several high-tech weapons systems. Among them was a portable radar that detected enemy artillery fire so quickly that return fire was often in the air before the enemy projectile landed. Basically, the radar coordinatizes the plane in which the projectile is traveling, samples several points in the projectile's path, fits a model to the points, and uses the model to determine the source of the fire. Since the system is computerized, it all happens very quickly.

   Suppose the radar, which is at the origin of the coordinate system, detects an artillery shell headed in its direction. The coordinates of three points in the projectile's path are (200, 25.4), (252.63, 30.16), and (300, 20.63), as shown in **Figure 3.2**.

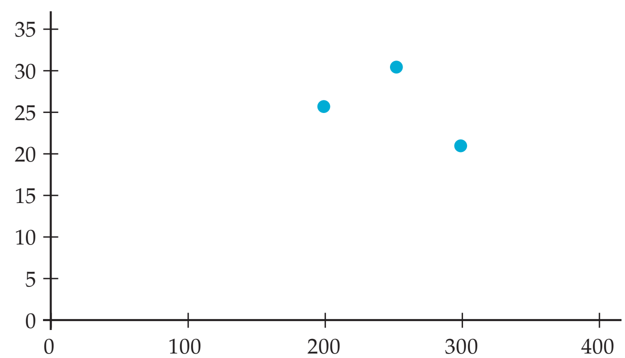


FIGURE 3.2.

   a) Explain why a quadratic function would be a good model in this context.

   b) Write a system of equations for the coefficients of a quadratic model.

   c) Using two different methods, find a quadratic model for these data.

   d) How well does your model fit these data? Explain.

   e) Use your model to determine the coordinates of the projectile's source and its point of impact.

Exercises 3.2

2. Surveyors use compasses to do their work. Unfortunately, compasses point to the magnetic pole, not the true pole, and the magnetic pole is not stationary. That means that surveys conducted many years ago must be rotated to fit modern maps. To determine the proper amount of rotation, it is necessary to know the declination, which is the difference (measured in degrees) between true north and magnetic north, at the time of the survey and at the present (see **Figure 3.3**).

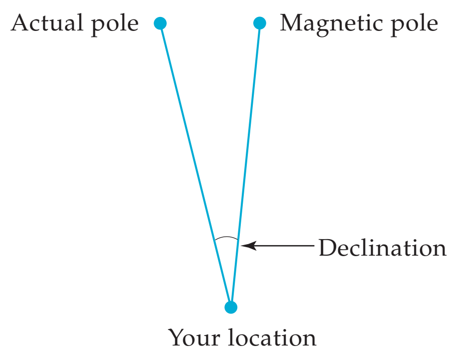


FIGURE 3.3.

Year	Declination
1800	2° 48' East
1810	2° 56' East
1820	2° 53' East
1830	2° 42' East
1840	2° 21' East
1850	1° 54' East
1860	1° 21' East
1870	0° 44' East
1880	0° 4' East
1890	0° 36' West
1900	1° 11' West
1910	1° 39' West
1920	2° 2' West
1930	2° 30' West
1940	2° 31' West
1950	2° 25' West
1960	2° 41' West
1970	3° 25' West
1980	4° 45' West
1990	5° 52' West
2000	6° 58' West

TABLE 3.5.

Declination varies with time and location, but the factors that cause changes in declination are not well understood. (Information on declination is available from the National Geophysical Data Center at their web site.) **Table 3.5** gives declination for Charleston, West Virginia at 10-year intervals.

a) Analyze the data and use your tool kit of functions to develop a model that you feel would be appropriate to use to estimate the declination at Charleston between 1800 and 2000. (Suggestions: Convert degrees and minutes to decimals and associate negative with either east or west, and positive with the other.)

b) Would you advise the use of your model to predict declinations before 1800 or after 2000? Explain.

3. Consider the data in **Table 3.6**. How well does the polynomial function $y = 0.0018x^4 - 0.205x^3 + 7.78x^2 - 121x + 740$ capture the trend in the data? Explain. If possible, find a better quartic (fourth-degree) polynomial model.

x	y
12	88.4
15	71.7
19	76.5
22	83.7
28	72.3

TABLE 3.6.

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x	y
7.4	171
8.1	170
8.9	166
9.8	160
10.1	158
10.4	154

TABLE 3.7.

4. Evaluate the cubic $y = -1.56x^3 + 39.22x^2 - 330.4x + 1101.3$ as a model for the data in **Table 3.7.**

5. One characteristic of polynomial functions that makes them useful modeling tools is that they can capture peaks and valleys in data.

   a) Experiment with various polynomial functions and their graphs. What can you conclude about the relationship between the degree of a polynomial and the number of peaks and valleys it can model? (For example, can a third-degree polynomial's graph have four peaks/valleys?)

   b) You can form a quadratic function that has zeros at 1 and 5 by multiplying the linear factors $(x - 1)$ and $(x - 5)$. How are the zeros related to the peak/valley of the quadratic's graph?

   c) One way to form a polynomial of degree n is to multiply n linear factors (factors of the form $ax - b$). How does the number of linear factors a polynomial can have support the answer you gave in (a)?

6. Use symbolic methods or a calculator program to solve the systems of equations in (a) and (b).

   a) $$\begin{aligned} 2x + 3y + 11 &= 0 \\ 5x - 4y - 30 &= 0 \end{aligned}$$

   b) $$\begin{aligned} x - 3y + 2z + 2 &= 0 \\ 2x + 3y - 4z + 7 &= 0 \\ x - y + 6z - 6 &= 0 \end{aligned}$$

   c) Some systems of equations have no solutions. Use a graph to explain why this system has no solution.

   $$\begin{aligned} 3x - 2y + 5 &= 0 \\ 6x - 4y - 9 &= 0 \end{aligned}$$

7. For each of the polynomial functions (a)–(c), build a table of values for $x = 1$ through 6. Investigate the rate at which y changes by including columns of first differences, second differences, and so forth until the differences become constant. What can you conclude about rates of change of polynomial functions? (If necessary, experiment with a few additional polynomial functions of your own.) How are differences related to the degree of a polynomial?

a) $y = x^2 + 2x - 1$.

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b) $y = 2x^3 - 4x^2 - 3$.

c) $y = x^4 - 2x^3 - 5x^2 + 1$.

8. If you drop all but the first term of the polynomial function $y = 2x^4 - 3x^3 + 4x^2 - 7$, you have a power function. Compare the graphs of the polynomial function and the associated power function in larger and larger windows. (Use your calculator's zoom out feature to do this quickly.) Do the same for other polynomial functions and the corresponding power functions obtained by dropping all terms except the one with the highest power. What can you conclude about the end behavior of a polynomial and its related power function?

Background for Exercises 9 and 10. The ability of polynomial functions to pass through every point in a data set has many important applications. One of them is in the field of error-correction codes—codes that correct errors caused by static or other kinds of noise. These codes are used to transmit satellite photos to Earth and to allow you to listen to a scratched compact disc. The codes do their work by representing photos and music as a collection of numbers and adding enough additional data to the message to make recovery of corrupted data possible.

The most common type of polynomial error-correcting code is known as a Reed-Solomon code. In practice, Reed-Solomon codes work with binary numbers—0s and 1s—and often encode several hundred numbers at a time, which requires a polynomial of very high degree.

The error correction in your compact disc player is probably done with a Reed-Solomon code. That's mathematics—music to your ears!

9. Consider a simple example. Suppose you want to send a three-number code composed of the values 5.2, 6.8, and 7.9, in that order. A quadratic can be passed through all three pairs (1, 5.2), (2, 6.8), and (3, 7.9), and quadratic regression gives the model $y = -0.25x^2 + 2.35x + 3.1$. To avoid errors that might occur in transmission, add several new values to the information to be coded by evaluating the polynomial for, say, 4, 5, 6, 7, 8, 9, and 10. (The actual number depends on the probability of error—the higher the probability, the more extra information you need.) Together with the new information, you now have the numbers 5.2, 6.8, 7.9, 8.5, 8.6, 8.2, 7.3, 5.9, 4, 1.6.

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All ten of these numbers are transmitted in the order in which they are listed. Suppose that two of them, the third and the sixth, have errors, and the transmission comes through as 5.2, 6.8, 9.9, 8.5, 8.6, 6.2, 7.3, 5.9, 4, 1.6. The decoding device, which knows that a quadratic model was used but not the specific quadratic, detects that these two do not fit a quadratic model. The decoding device determines a quadratic model for the remaining eight, and uses the model to correct the two that are in error.

a) Explain why the two data with errors do not fit a quadratic model.

b) Show how a quadratic model can be found for the other eight.

c) Show how to use the model to correct the errors.

10. a) Show how to use a polynomial function to encode the numbers 2.5, 4.9, 1.6, and 6.8.

b) Encode four extra values.

11. **Investigation.** Return to the data of Activity 3.2. The data appear to have been rounded to three significant digits. Thus, for example, the value recorded as 400,000 feet may actually have been anywhere between 400,500 feet and 399,500 feet. Similar statements can be made for the other values. Rework Items 1 and 2 of Activity 3.2 using different combinations of possible actual values for the heights, and comment on the implications for the predictions you made in the activity.